EXERCISESET 5, TOPOLOGY IN PHYSICS

- The hand-in exercise is the exercise 1.
- Please hand it in electronically at `topologyinphysics2019@gmail.com` (1 pdf!)
- Deadline is Wednesday March 20, 23.59.
- Please make sure your name and the week number are present in the file name.

Exercise 1: The group of gauge transformations. In the lectures we have seen that gauge theories, such as Yang–Mills theory, are described using principal G-bundles for some fixed compact Lie group G. These theories are called gauge theories because they have a large symmetry group, the *group of gauge transformations*. The aim of this exercise is to better understand the structure of this group.

Fix a principal G-bundle $\pi\colon P \to M$. The gauge group is defined to be

$$\mathcal{G}(P) := \left\{ \psi\colon P \to P, \text{ satisfying } \begin{matrix} \psi(pg) = \psi(p)g \\ \pi(\psi(p)) = \pi(p) \end{matrix} \text{ for all } p \in P,\ g \in G \right\},$$

where all ψ are assumed to be smooth.

a) Show that $\mathcal{G}(P)$ is indeed a group. When showing that $\mathcal{G}(P)$ contains inverses you may assume that they are smooth.

b) Suppose that $P = M \times G$ is the trivial bundle. Show that the equation

$$\psi(m,g) = (m, \varphi(m)g) \quad \text{for all } (m,g) \in P,$$

defines, for any $\psi \in \mathcal{G}(P)$, a function $\varphi : M \to G$. Explain that this correspondence shows that $\mathcal{G}(P) \cong C^\infty(M,G)$. What is the group structure on $C^\infty(M,G)$?

c) Let A be a connection form on P, i.e. $A \in \Omega^1(P,\mathfrak{g})$ is a Lie algebra valued 1-form that satisfies

$$\iota_{\xi_P} A = \xi, \quad \text{for all } \xi \in \mathfrak{g},$$
$$R_g^* A = \mathrm{Ad}_{g^{-1}}(A), \quad \text{for all } g \in G.$$

Show that for $\psi \in \mathcal{G}(P)$, the pull-back $\psi^* A$ is another connection 1-form.

d) When P is trivial, it has a global section $s : M \to P$ and we can use this section to pull the connection form A back to a $\mathfrak{g}$-valued 1-form $\alpha := s^* A \in \Omega^1(M,\mathfrak{g})$. A computation shows that the action of $\mathcal{G}(P)$ on the space of connections is given by

$$\varphi \cdot \alpha = \varphi^{-1}\alpha\varphi + \varphi^{-1} d\varphi, \tag{1}$$

using the notation of b). Show by an explicit computation that this defines an action of $\mathcal{G}(P)$: $\varphi_1 \cdot (\varphi_2 \cdot \alpha) = (\varphi_2\varphi_1) \cdot \alpha$.

e) Show that the curvature $F(\alpha) := d\alpha + \alpha \wedge \alpha$ satisfies

$$F(\varphi \cdot \alpha) = \varphi^{-1} F(\alpha) \varphi.$$

Use this to show that the Yang–Mills action

$$S_{YM}(\alpha) := \int_M \mathrm{Tr}(F(\alpha) \wedge \star F(\alpha))$$

is invariant under the action of the gauge group.

Exercise 2: Cocycles and Representations. In this exercise we will consider how one uses representations of general linear groups in order to perform constructions from linear algebra like the direct sum and tensor product. For this purpose recall that a rank n vector bundle $E \to M$ and a cover $\cup_\alpha U_\alpha = M$ yield a cocycle

$$\varphi_{\alpha\beta} \colon U_\alpha \cap U_\beta \longrightarrow GL(n)$$

and we may reconstruct the vector bundle E given the cocycle. Now suppose $\phi \colon GL(n) \to GL(m)$ is a group homomorphism, i.e. a representation of $GL(n)$ on the m-dimensional standard vector space. In particular $\phi(I) = I$, $\phi(A^{-1}) = \phi(A)^{-1}$ and $\phi(AB) = \phi(A)\phi(B)$ and so $\phi \circ \varphi_{\alpha\beta}$ satisfies the cocycle conditions. Thus the representation ϕ coupled with the vector bundle E define another (rank m) vector bundle S!

i) Recall that $E \oplus E$ is the vector bundle with fiber $E_x \oplus E_x$. Give the representation $GL(n) \to GL(2n)$ that gives rise to this vector bundle.
ii) Similarly $E \otimes E$ is the vector bundle with fiber $E_x \otimes E_x$. Give the representation $GL(n) \to GL(n^2)$ that gives rise to this vector bundle.
iii) Again $\bigwedge^n E$ is the vector bundle with fiber $E_x \wedge E_x \wedge \ldots \wedge E_x$ (n-times). Give the representation $GL(n) \to GL(1)$ that gives rise to this vector bundle.
iv) The vector bundle E^* is the vector bundle with fibers E_x^*. Give the representation $GL(n) \to GL(n)$ that gives rise to this vector bundle.

Given another rank k vector bundle F with corresponding cocycle denoted by $\psi_{\alpha\beta} \colon U_{\alpha\beta} \to GL(k)$ we find that a representation of the product $\phi \colon GL(n) \times G(k) \to GL(m)$ gives rise to the cocycle

$$\phi \circ (\varphi_{\alpha\beta}, \psi_{\alpha\beta}) \colon U_{\alpha\beta} \longrightarrow GL(m)$$

and thus a rank m vector bundle S.

i) Recall that $E \oplus F$ is the vector bundle with fiber $E_x \oplus F_x$. Give the representation $GL(n) \to GL(n+k)$ that gives rise to this vector bundle.
ii) Similarly $E \otimes F$ is the vector bundle with fiber $E_x \otimes F_x$. Give the representation $GL(n) \to GL(nm)$ that gives rise to this vector bundle.
iii) Again $Hom(E, F)$ is the vector bundle with fiber $Hom(E_x, F_x)$. Give the representation $GL(n) \to GL(nm)$ that gives rise to this vector bundle. What does this have to do with the vector bundle $E^* \otimes F$?

Exercise 3: Non-trivial subbundle. In this exercise we will consider the relation between projection valued functions and vector bundles. In the proces we will construct a non-trivial subbundle of a trivial bundle. In fact on a compact manifold every bundle is a subbundle of a trivial one since any vector bundle E in that case admits a *complementary* bundle F such that $E \oplus F$ is trivial. We will consider the base space $M = \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, the 2-dimensional torus, and we will denote the rank 2 trivial bundle $\mathbb{T}^2 \times \mathbb{C}^2$ by $C^2 \to \mathbb{T}^2$.

i) Show that the sections $\Gamma^\infty(\mathbb{T}^2; C^2)$ are given by columns $\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$ of smooth functions $\eta_i \in C^\infty(\mathbb{T}^2)$.

ii) Show that the map $D\colon \Gamma^\infty(\mathbb{T}^2; C^2) \to \Omega^1(\mathbb{T}^2; C^2)$ given by

$$D \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} d\eta_1 \\ d\eta_2 \end{pmatrix}$$

defines a connection on C^2.

iii) Show that the curvature $F(D)$ vanishes identically.

Suppose f, g and h are smooth functions on the circle such that

$$f - f^2 = g^2 + h^2 \quad \text{and} \quad gh = 0.$$

iv) Consider the smooth function $p\colon \mathbb{T}^2 \to M_2(\mathbb{C})$ given by

$$p(\theta, \phi) = \begin{pmatrix} f(\theta) & g(\theta) + h(\theta)e^{i\phi} \\ g(\theta) + h(\theta)e^{-i\phi} & 1 - f(\theta) \end{pmatrix}$$

and show that $p^2 = p$.

Now we note p defines a map $P\colon C^2 \to C^2$ that is linear in the fibers and since $p^2 = p$ the image $\operatorname{Im} P$ defines a rank 1 vector bundle $N^1 \to \mathbb{T}^2$.

v) Show that map $\nabla\colon \Gamma^\infty(\mathbb{T}^2; N^1) \to \Omega^1(\mathbb{T}^2; N^1)$ given by

$$\nabla P\eta = PDP\eta$$

defines a connection on N^1.

Recall that we may extend the definition of ∇ to a map

$$\nabla\colon \Omega^k(\mathbb{T}^2; N^1) \longrightarrow \Omega^{k+1}(\mathbb{T}^2; N^1)$$

in the obvious way, i.e. by the formula $PD = \nabla$ and noting that D is extended as the entrywise exterior derivative. Then the curvature $F(\nabla)$ is defined by the equation $\nabla^2 P\eta = F(\nabla) \wedge P\eta$.

vi) Show that $F(\nabla) = pdp \wedge dp$

In a following lecture we will see that $Tr(F(\nabla))$ is a closed 2-form whose class is called the *first Chern class* (up to a factor). We will also see that this class only depends on the isomorphism class of the vector bundle. Moreover we will see that the first Chern class of a trivial vector bundle always vanishes. In this case we may compute that $Tr(F(\nabla))$ is not exact by integrating it over $\mathbb{T}^2$ and therefore N^1 is a non-trivial subbundle of the trivial bundle C^2. As a last (very hard) exercise you can try to show that $Tr(F(\nabla))$ is not exact by assuming the following 4 conditions on f, g and h

(1) $0 \leq f(\theta) \leq 1$ for all θ;
(2) $f(0) = 1$ and $f(\pi) = 0$;
(3) $g(\theta) = \sqrt{f - f^2}$ and $h(\theta) = 0$ for all $\theta \in [0, \pi]$ and finally
(4) $g(\theta) = 0$ and $h(\theta) = \sqrt{f - f^2}$ for all $\theta \in [\pi, 2\pi]$.